

ON AN ENERGY EQUALITY IN THE THEORY OF EVOLUTION EQUATIONS

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ABSTRACT. We give an elementary proof of a basic “energy” equality in the theory of evolution equations in the setting, which usually starts with introducing $V \subset H = H^* \subset V^*$.

1. MAIN RESULT, DISCUSSION, AND EXAMPLES

Let V be a Banach space over the field of complex numbers and let H be a Hilbert space over the field of complex numbers with norms $\|\cdot\|_V$, $\|\cdot\|_H$ and scalar product $(\cdot, \cdot)_H$, respectively.

Assumption 1.1. We have $V \subset H$ and for $v \in V$, $\|v\|_V \geq \|v\|_H$.

Let V^* be the dual space to V with pairing between them denoted by $\langle v^*, v \rangle$, $v \in V$, $v^* \in V^*$. Fix $T \in (0, \infty)$ and $p \in [1, \infty)$.

Assumption 1.2. For $t \in [0, T]$, we are given a V -valued strongly (Lebesgue) measurable function v_t and a V^* -valued function v_t^* such that

- (i) $\langle v_t^*, v \rangle$ is a measurable function of $t \in [0, T]$ for any $v \in V$,
- (ii) there is a measurable function f_t such that $\|v_t^*\|_{V^*} \leq f_t$ for $t \in [0, T]$,
- (iii) there is a constant $N_0 \in [1, \infty)$ such that

$$\begin{aligned} \int_0^T (\|v_t\|_V^p + f_t^{p'}) dt &\leq N_0, \quad p' = p/(p-1), \quad \text{if } p > 1, \\ \int_0^T \|v_r\|_V dr + f_t &\leq N_0 \quad (\text{a.e.}) \quad t \in [0, T] \quad \text{if } p = 1. \end{aligned} \quad (1.1)$$

Assumption 1.3. For any $v \in V$, for almost all $s, t \in [0, T]$, we have

$$(v_t, v)_H - (v_s, v)_H = \int_s^t \langle v_r^*, v \rangle dr. \quad (1.2)$$

Our main goal is to present an elementary proof of the following result.

Theorem 1.4. *Under the above assumptions there exists an H -valued H -strongly continuous function u_t , $t \in [0, T]$, such that $u_t = v_t$ for almost all $t \in [0, T]$, and for all $t \in [0, T]$ and $v \in V$*

$$(u_t, v)_H = (u_0, v)_H + \int_0^t \langle v_s^*, v \rangle ds. \quad (1.3)$$

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Furthermore, for all $t \in [0, T]$

$$\|u_t\|_H^2 = \|u_0\|_H^2 + 2 \int_0^t \Re \langle v_s^*, u_s \rangle ds. \quad (1.4)$$

Corollary 1.5. *Under the above assumption, suppose that V is dense in H (in the metric of H) and we are given $u \in H$ such that for any $v \in V$*

$$(v_t, v)_H = (u, v)_H + \int_0^t \langle v_r^*, v \rangle dr$$

for almost all $t \in [0, T]$. Then, of course, (1.2) holds for almost all $s, t \in [0, T]$ as long as $v \in V$, there exists u_t with the properties described in the theorem, and since $(u_0, v)_H = (u, v)_H$ for any $v \in V$ and V is dense in H , we have $u_0 = u$.

Remark 1.6. Obvious versions of Theorem 1.4 and Corollary 1.5 are true when V and H are spaces over the field of real numbers. This will be easily seen from the proof of Theorem 1.4.

The results like Theorem 1.4 and Corollary 1.5 are widely used in the theory of nonlinear (quasi-linear) parabolic partial differential equations satisfying a monotonicity condition. These results serve as an intermediate tool to achieve the goal of proving existence and uniqueness of solutions. Probably because of that the author could not find Theorem 1.4 and Corollary 1.5 in the above generality in the literature.

It seems the results like Theorem 1.4 and Corollary 1.5 first appeared in J.-L. Lions [3] when V was a Hilbert space. However, the energy equality (1.4) is not singled out because there were just no point in proving it. A different proof of the existence of H -continuous modifications using mollification in t is given in [4]. There the energy equality (1.4) is used as an obvious fact.

Then in Remark 2.1.2 on page 156 of J.-L. Lions [5] we read: Let V be a reflexive Banach space embedded into a Hilbert space H , $V \subset H$, with continuous embedding, and let V be dense in H ; after identifying H with its dual and denoting V^* the dual of V , one can identify H with a subspace of V^* , so that $V \subset H \subset V^*$. Then, if we are given a function $u \in L_p((0, T), V)$ such that $du/dt \in L_{p'}((0, T), V^*)$, the function u is (after being modified on a set of measure zero) a strongly continuous H -valued function on $[0, T]$ and the mapping $u \rightarrow u(0)$ is onto H .

No further comments, references, or proofs are given, and in the author's opinion, for an inexperienced reader even to understand what exactly is claimed is a challenge. Is it assumed that $p > 2$ as in the title of the subsection 2.1.1? Of course, the above statement in [5] is just a side remark and cannot be regarded as a flaw of this book remarkable in all respects.

Similar situation we have with Remark 2.7.9 on page 236 of J.-L. Lions [5], where $V = \overset{0}{W}_p^1(\Omega) \not\subset H = L_2(\Omega)$ (if $p \leq 2d/(d+2)$, $d = \dim(\Omega)$), $u \in L_p((0, T), V)$ and $du/dt \in L_{p'}((0, T), V^*)$. Still the claim is that the function u is (after being modified on a set of measure zero) a strongly

continuous H -valued function on $[0, T]$. Again no comments or proofs are given and how the trivial example of $u_t \equiv v \in V \setminus H$ is excluded is not clear to an inexperienced reader. We show a way to treat such situations in an almost trivial generalization of Corollary 1.5 in Section 3.

The only place where the author could find the proof of the results like Theorem 1.4 and Corollary 1.5 when V is *not* a Hilbert space is the article by F.E. Browder [1], where V is a closed subset of W_p^m , $p \in (1, \infty)$, hence a reflexive space. He considers the closure of the operator d/dt as an operator from $X = L_p(\mathbb{R}, V)$ to X^* . However, his setting is quite different from the one based on $V \subset H = H^* \subset V^*$.

We decided to only concentrate on Theorem 1.4 and Corollary 1.5 and prove them in the most general setting requiring only basic knowledge of functional analysis to understand the statements and their rather elementary proofs. We do not assume that V is reflexive, or dense in H , or separable. Our main tools are some standard arguments from measure theory, Doob's remarkable theorem on approximation of Lebesgue integrals by Riemannian sums, and the following two formulas

$$(u_2, v_2)_H - (u_1, v_1)_H = (u_2 - u_1, v_1)_H + \overline{(v_2 - v_1, u_2)_H}$$

$$\|u_2\|_H^2 - \|u_1\|_H^2 = 2\Re(u_2 - u_1, u_1)_H + \|u_2 - u_1\|_H^2 \geq 2\Re(u_2 - u_1, u_1)_H. \quad (1.5)$$

The proofs given here are adaptations of the proof of Itô's formula for Banach space-valued stochastic processes from [2], where $p \in (1, \infty)$. Unlike [4] and [1], in the stochastic setting approximating v_t with functions smoother in time does not lead to anything reasonable and one could only use the tools mentioned above. It is worth noting that a different proof of Itô's formula for Banach space-valued stochastic processes is given earlier in [6]. This proof is based on Remark 2.1.2 on page 156 of J.-L. Lions [5].

We first give two examples of applications of Theorem 1.4 in two border situations, then prove this theorem in Section 2. As pointed out above, in Section 3 we present a generalization of the main theorem in the case that $V \not\subset H$.

Here is an example of application of Theorem 1.4 in which v_t^* is not strongly measurable as a V^* -valued function.

Example 1.7. For $q \in [1, \infty)$ let ℓ_q be the Banach space of sequences $u = (u_0, u_1, \dots)$ of complex numbers such that

$$\|u\|_{\ell_q} = \left(\sum_{n=0}^{\infty} |u_n|^q \right)^{1/q} < \infty,$$

and let ℓ_∞ be the Banach space of sequences such that

$$\|u\|_{\ell_\infty} = \sup_n |u_n| < \infty.$$

Introduce $V = \ell_1$, then $V^* = \ell_\infty$ with pairing

$$\langle v^*, v \rangle = \sum_{n=0}^{\infty} v_n^* \bar{v}_n.$$

Also introduce $H = \ell_2$ with the scalar product

$$(u, v) = \sum_{n=0}^{\infty} u_n \bar{v}_n.$$

Observe that $V \subset H$, $\|v\|_V \geq \|v\|_H$, and consider the function

$$v_t = (v_0(t), v_1(t), \dots), \quad v_n(t) = 2^{-n} \exp(i2^n \pi t), n = 0, 1, 2, \dots$$

Obviously, v_t is a V -valued strongly continuous, and hence measurable, function defined for $t \geq 0$ and

$$\int_0^T \|v_t\|_V^2 dt < \infty.$$

Furthermore, for any $v \in V$

$$\begin{aligned} (v_t, v)_H &= (v_0, v)_H + \sum_{n=0}^{\infty} 2^{-n} [\exp(i2^n \pi t) - 1] \bar{v}_n \\ &= (v_0, v)_H + i\pi \sum_{n=0}^{\infty} \int_0^t \exp(i2^n \pi s) \bar{v}_n ds \\ &= (v_0, v)_H + i\pi \int_0^t \sum_{n=0}^{\infty} \exp(i2^n \pi s) \bar{v}_n ds = (v_0, v)_H + \int_0^t \langle v_s^*, v \rangle ds, \end{aligned}$$

where

$$v_t^* = (v_0^*(t), v_1^*(t), \dots), \quad v_n^*(t) = i\pi \exp(i2^n \pi t), n = 0, 1, 2, \dots$$

Obviously, v_t^* is a V^* -valued function, $\|v_t^*\|_{V^*} \leq \pi =: f_t$,

$$\int_0^T f_t^2 dt < \infty,$$

and for any $v \in V$, $\langle v_t^*, v \rangle$ is measurable. By Theorem 1.4 with $p = 2$

$$\|v_t\|_H^2 = \sum_{n=0}^{\infty} 2^{-2n} = \|v_0\|^2 + 2\Re \int_0^t \sum_{n=0}^{\infty} 2^{-n} i\pi ds = \|v_0\|^2.$$

This result is, of course, trivial in itself. What is remarkable though is that v_t^* is not a strongly measurable V^* -valued function, and condition (i) in Assumption 1.2 is essential.

To prove this fact, it suffices to show that the set of values of v_t^* in V^* as t belongs to any set of full measure in $(0, T)$ is nonseparable. We claim that a stronger assertion holds: for any $s, t \in [0, 1]$, such that $s \neq t$, we have

$$\sup_{n=0,1,2,\dots} |\exp(i2^n \pi t) - \exp(i2^n \pi s)| \geq |1 - \exp(i2\pi/3)|. \quad (1.6)$$

Assume that (1.6) is false, set $r = t - s$, and suppose without loss of generality that $r > 0$. Then, for any $n = 0, 1, 2, \dots$, there exists an integer N_n and a number ϕ_n such that

$$2^n \pi r = 2N_n \pi + \pi \phi_n, \quad |\pi \phi_n| < 2\pi/3.$$

In short

$$2^n r = 2N_n + \phi_n, \quad |\phi_n| < 2/3.$$

Since $r > 0$, $N_0 = 0$ and $\phi_0 = r > 0$. Then for

$$n_0 = \left[\log_2 \frac{2}{3\phi_0} \right] + 1$$

where $[a]$ is the integer part of a , we have $2^{n_0} r = 2^{n_0} \phi_0$ and

$$2/3 < 2^{n_0} \phi_0 \leq 4/3$$

since $a - 1 < [a] \leq a$. It follows that in the representation $2^{n_0} r = 2N_{n_0} + \phi_{n_0}$ ($= 2^{n_0} \phi_0$) with smallest possible $|\phi_{n_0}|$, either $N_{n_0} = 0$ and $\phi_{n_0} = 2^{n_0} \phi_0 > 2/3$ or $N_{n_0} = 1$ and $\phi_{n_0} = 2^{n_0} \phi_0 - 2 \leq -2/3$. This is the desired contradiction proving our claim.

Here is an example where again V is non reflexive and $p = 1$.

Example 1.8. For $n = 1, 2, \dots$ set $c_n = 2^{-n}$ and for $x \in I := (-\pi, \pi)$ and $t \in [c_{n+1}, c_n]$ define

$$k_n = [\ln_2 n], \quad a_n = 2^{n-k_n},$$

$$v_t(x) = (2^{n+1}t - 1)c_n \sin(a_n x) + (2 - t2^{n+1})c_{n+1} \sin(a_{n+1}x).$$

Observe that u_t is obtained by the linear interpolation on $t \in [c_{n+1}, c_n]$ between the values at the end points, which are $c_{n+1} \sin(a_{n+1}x)$ and $c_n \sin(a_n x)$. The function $v_t(x)$ is defined in $I \times (0, T)$, where $T = 1/2$.

Note that

$$\int_{-\pi}^{\pi} |D^2 v_{c_n}| dx = c_n a_n^2 \int_{-\pi}^{\pi} |\sin(a_n x)| dx,$$

which is of order 2^{n-2k_n} as $n \rightarrow \infty$. The series $\sum 2^{-2k_n}$ converges and this implies that $v \in L_1((0, T), V)$, where $V = L_2(I) \cap \overset{0}{W}_1^2(I)$ is a Banach space with norm defined as the sum of norms in $L_2(I)$ and $\overset{0}{W}_1^2(I)$. Obviously, $V \subset H := \overset{0}{W}_2^1(I)$.

Finally, the first derivative of $v_t(x)$ with respect to t , say $v_t^*(x)$, is a bounded function, so that $v_t^* \in V^*$, and for $v \in C^2(I)$ vanishing at $\pm\pi$, by Fubini's theorem we have

$$(v_t, v)_H = \int_{-\pi}^{\pi} v_t(1 - D^2 v) dx = \int_0^t \langle v_s^*, v \rangle ds,$$

where

$$\langle v_s^*, v \rangle = \int_{-\pi}^{\pi} v_s^*(1 - D^2 v) dx$$

extends to a bounded linear functional on V with its norm bounded on $(0, T)$. By Corollary 1.5 the function v_t extends to $[0, T]$ as a strongly continuous H -valued function equal to zero at $t = 0$.

This is not a surprising result of course. However, observe that it follows from

$$\int_{-\pi}^{\pi} |D^2 v_t| dx \geq (2^{n+1}t - 1) \int_{-\pi}^{\pi} |D^2 v_{c_n}| dx - (2 - t2^{n+1}) \int_{-\pi}^{\pi} |D^2 v_{c_{n+1}}| dx, \quad (1.7)$$

valid for $t \in [c_{n+1}, c_n)$, that there exists an absolute constant $\varepsilon \in (0, 1/2)$ such that if $t \in [c_n(1 - \varepsilon), c_n)$, then the right-hand side of (1.7) is bigger than $\varepsilon 2^{n-2k_n}$. An easy consequence of this is that $v \notin L_p((0, T), \dot{W}_q^2(I))$ for $p, q \geq 1$ unless $p = q = 1$.

2. PROOF OF THEOREM 1.4

The proof of Theorem 1.4 is based on three lemmas. The first one is just a reminder of part of basics of integration theory.

Lemma 2.1. *Let F be a Banach space and let f_t , $t \in [0, T]$, be a strongly measurable F -valued function such that for a $q \in [1, \infty)$ we have*

$$\int_0^T \|f_t\|_F^q dt < \infty.$$

Set $f_t = 0$ for $t < 0$ and $t > T$. Then

$$I := \lim_{a \rightarrow 0} \int_0^T \|f_t - f_{t+a}\|_F^q dt = 0.$$

Proof. By definition, for any $\varepsilon > 0$ there exists an integer n , measurable sets $A_k \subset [0, T]$, and $g_k \in F$, $k = 1, \dots, n$, such that, for

$$\phi_t := \sum_{k=1}^n g_k I_{A_k}(t),$$

we have

$$\int_0^T \|f_t - \phi_t\|_F^q dt \leq \varepsilon.$$

Also, it is known that for any measurable sets $A \subset [0, 1]$

$$\lim_{a \rightarrow 0} \int_0^T \|I_A(t) - I_A(t+a)\|^q dt = 0.$$

By also taking into account that

$$\int_0^T \|f_{t+a} - g_{t+a}\|_F^q dt \leq \int_0^T \|f_t - g_t\|_F^q dt,$$

we find

$$\overline{\lim}_{a \rightarrow 0} \int_0^T \|f_t - f_{t+a}\|_F^q dt \leq 2 \int_0^T \|f_t - \phi_t\|_F^q dt \leq 2\varepsilon,$$

and the lemma is proved.

Denote $\kappa_{(-)}(n, x) = 2^{-n}[2^n x]$, $\kappa_{(+)}(n, x) = 2^{-n}[2^n x] + 2^{-n}$, where $[x]$ is the integer part of $x \in \mathbb{R}$, $n = 1, 2, \dots$

The following result, basically, belongs to J. Doob.

Lemma 2.2. *Let the assumption of Lemma 2.1 be satisfied.*

Set $f_t = 0$ for $t < 0$ and $t > T$. Then there exists a sequence of integers n_k , $k = 1, 2, \dots$, such that $n_k \rightarrow \infty$ and for almost any $c \in (0, 1)$

$$\int_0^T \|f_t - f_{\kappa_{(\pm)}(n_k, t+c)-c}\|_F^q dt \rightarrow 0 \quad (2.1)$$

as $k \rightarrow \infty$.

Proof. The function $\|f_t - f_s\|_F$ is a measurable function of (t, s) . Hence, $\|f_t - f_{\kappa_{(\pm)}(n, t+c)-c}\|_F$ are measurable as well for either sign $+$ or $-$. By Fubini's theorem

$$\int_0^1 \int_0^T \|f_t - f_{\kappa_{(\pm)}(n, t+c)-c}\|_F^q dt dc = \int_0^T \Phi_{\pm n}(t) dt,$$

where

$$\begin{aligned} \Phi_{\pm n}(t) &:= \int_0^1 \|f_t - f_{t+\kappa_{(\pm)}(n, t+c)-(t+c)}\|_F^q dc = \int_t^{t+1} \|f_t - f_{t+\kappa_{(\pm)}(n, c)-c}\|_F^q dc \\ &= \int_0^1 \|f_t - f_{t+\kappa_{(\pm)}(n, c)-c}\|_F^q dc, \end{aligned}$$

where the last equality follows from the fact that $\|f_t - f_{t+\kappa_{(\pm)}(n, c)-c}\|_F$ are periodic functions of c and one of their periods equals one.

Hence,

$$\int_0^1 \int_0^T \|f_t - f_{\kappa_{(\pm)}(n, t+c)-c}\|_F^q dt dc = \int_0^1 \Psi_{\pm n}(c) dc, \quad (2.2)$$

where

$$\Psi_{\pm n}(c) = \int_0^T \|f_t - f_{t+\kappa_{(\pm)}(n, c)-c}\|_F^q dt.$$

Observe that $\Psi_{\pm n}(c) \rightarrow 0$ as $n \rightarrow \infty$ for any $c \in [0, 1]$ by Lemma 2.1. Furthermore, obviously

$$|\Psi_{\pm n}(c)| \leq 2^q \int_0^T \|f_t\|_F^q dt.$$

Hence in light of (2.2) and the dominated convergence theorem

$$a(n, c) := \int_0^T \left(\|f_t - f_{\kappa_{(-)}(n, t+c)-c}\|_F^q + \|f_t - f_{\kappa_{(+)}(n, t+c)-c}\|_F^q \right) dt \rightarrow 0$$

in $L_1([0, 1])$. Then there is a subsequence $n_k \rightarrow \infty$ such that $a(n_k, c) \rightarrow 0$ for almost any $c \in (0, 1)$. This is exactly what is asserted and the lemma is proved.

Remark 2.3. Observe that

$$x \geq \kappa_{(-)}(n, x) \geq x - 2^{-n}, \quad x \leq \kappa_{(+)}(n, x) \leq x + 2^{-n},$$

the functions $\kappa_{(\pm)}(t)(n, x)$ are right-continuous piece-wise constant and have jumps at points $k2^{-n}$, $k = 0, \pm 1, \dots$. Also it is useful to note that, for a fixed $c \in (0, 1)$, the graph of the function $y = \kappa_{(-)}(n, x + c) - c$, $x \in \mathbb{R}$, is obtained from the graph of $y = \kappa_{(-)}(n, x) - x$, $x \in \mathbb{R}$, by sliding the latter appropriately along the diagonal $y = x$ in the direction of lesser values of the coordinates.

Next, if we have a set $C \subset (0, T)$ of full measure, define $D = C \cup (-\infty, 0) \cup (T, \infty)$ and observe that for any $k = 0, 1, 2, \dots$ and $n = 1, 2, \dots$ the point $k2^{-n} - c$ belongs to D for almost any $c \in (0, 1)$. It follows that there exists a set C_0 of full measure in $(0, 1)$ such that, for any $c \in C_0$, all points $k2^{-n} - c$ are in D for all $k = 0, 1, 2, \dots$ and $n = 1, 2, \dots$, that is for any $c \in C_0$

$$(0, T) \cap \{k2^{-n} - c : k = 0, 1, 2, \dots, n = 1, 2, \dots\} \subset C$$

Of course, we can take and fix $c_0 \in C_0$ so that (2.1) holds as well.

Lemma 2.4. *Under the assumptions of Theorem 1.4 there exists a set $C \subset (0, T)$ of full measure such that the set $v_C := \{v_t : t \in C\}$ is separable in the metric of V and equation (1.2) holds for any $v \in v_C$ and $s, t \in C$. Furthermore, if the sequence n_k is taken from Lemma 2.2 with $f_t = v_t$, $F = V$, and $q = p$, then there exists $c_0 \in (0, 1)$ such that (2.1) holds and all values of the functions*

$$\chi_{(\pm)}(k, t) := \kappa_{(\pm)}(n_k, t + c_0) - c_0.$$

which lie in $(0, T)$, belong to C for any k , and the same is true for all points of jumps of these functions (coinciding with the values of $\chi_{(-)}(k, t)$).

Proof. Since v_t is a strongly measurable V -valued function, there exists a set $A \subset (0, T)$ of full measure such that $v_A = \{v_t : t \in A\}$ is a separable set in V . Let $\{v^i, i = 1, 2, \dots\}$ be a countable subset of v_A dense in v_A in the metric of V .

Then observe that for each $v = v^i$ equation (1.2) holds for almost all $(s, t) \subset A \times A$. Since we have only countably many v^i 's, there exists a set $B \subset A \times A$ of full measure such that, for any $v = v^i$, equation (1.2) holds for all $(s, t) \in B$. By Fubini's theorem and in light of the fact that A has full measure, there exists $s_0 \in A$ such that set $C = \{t \in (0, T) : (s_0, t) \in B\}$, has full measure in $(0, T)$. Since $C \subset A$, $v_C \subset v_A$, and v_C is a separable subset of V .

Furthermore, by construction, for any $v \in \{v^i, i = 1, 2, \dots\}$ and $t \in C$ we have

$$(v_t, v)_H - (v_{s_0}, v)_H = \int_{s_0}^t \langle v_r^*, v \rangle dr.$$

By subtracting such equalities we see that (1.2) holds for any $v \in \{v^i, i = 1, 2, \dots\}$ and $s, t \in C$. Now since $\{v^i, i = 1, 2, \dots\}$ is dense in v_A and, say

$$\left| \int_s^t \langle v_r^*, v \rangle dr - \int_s^t \langle v_r^*, v^i \rangle dr \right| \leq \|v - v^i\|_V \int_0^T f_s ds,$$

it follows that equation (1.2) holds for all $s, t \in C$ and $v \in v_A = \{v_t : t \in A\}$ and, hence, for all $v \in v_C$ as well. This proves the first statement of the lemma. The second one follows directly from Remark 2.3. The lemma is proved.

Proof of Theorem 1.4. Take $\chi_{(\pm)}(k, t)$ from Lemma 2.4 and introduce

$$I_k = (0, T) \cap \{i2^{-n_k} - c_0 : i = 1, 2, \dots\}, \quad \rho = \bigcup_{k=1}^{\infty} I_k,$$

so that I_k is the set of values of $\chi_{(-)}(k, t)$, $t > 0$, which lie in $(0, T)$, $I_k, \rho \subset C$, $\chi_{(-)}(k, t) = t$, $\chi_{(+)}(k, t) = t + 2^{-n_k}$ for $t \in I_k$.

Next, take $s, t \in I_k$ such that $s < t$ and s, t are neighbors in I_k ($t - s = 2^{-n_k}$). Then from Lemma 2.4, (1.2) and the fact that $v_s, v_t \in v_C$, because $s, t \in C$, we get that

$$\begin{aligned} \|v_t\|_H^2 - \|v_s\|_H^2 &= (v_t - v_s, v_s)_H + \overline{(v_t - v_s, v_t)}_H = \int_s^t \langle v_r^*, v_s \rangle dr + \int_s^t \overline{\langle v_r^*, v_t \rangle} dr \\ &= \int_s^t \langle v_r^*, v_{\chi_{(-)}(k, r)} \rangle dr + \int_s^t \overline{\langle v_r^*, v_{\chi_{(+)}(k, r)} \rangle} dr. \end{aligned} \quad (2.3)$$

By having in mind summing up such equalities as telescopic sums we obtain that, if $s, t \in \rho$ are such that $s < t$, then there exists an integer k_0 for which $s, t \in I_{k_0}$ and

$$\|v_t\|_H^2 - \|v_s\|_H^2 = \int_s^t \langle v_r^*, v_{\chi_{(-)}(k, r)} \rangle dr + \int_s^t \overline{\langle v_r^*, v_{\chi_{(+)}(k, r)} \rangle} dr \quad (2.4)$$

with $k = k_0$. Since obviously, I_k 's are nested (2.4) also holds for all $k \geq k_0$.

We now fix $s, t \in \rho$ such that $s < t$ and let $k \rightarrow \infty$ in (2.4). Observe that

$$\left| \langle v_r^*, v_{\chi_{(\pm)}(k, r)} \rangle - \langle v_r^*, v_r \rangle \right| \leq \|v_{\chi_{(\pm)}(k, r)} - v_r\|_V f_r$$

and the right-hand side goes to zero in $L_1((0, T))$ as $k \rightarrow \infty$ owing to Assumption 1.2 (iii) and Hölder's inequality in case $p > 1$. It follows that $\langle v_r^*, v_r \rangle$ is a measurable function. Also

$$\left| \int_s^t \langle v_r^*, v_{\chi_{(\pm)}(k, r)} \rangle dr - \int_s^t \langle v_r^*, v_r \rangle dr \right| \leq \int_s^t \|v_{\chi_{(\pm)}(k, r)} - v_r\|_V f_r dr$$

and the right-hand side goes to zero as $k \rightarrow \infty$ by the above. As a result we see that, if $s, t \in \rho$ are such that $s < t$, then

$$\|v_t\|_H^2 - \|v_s\|_H^2 = 2 \int_s^t \Re \langle v_r^*, v_r \rangle dr. \quad (2.5)$$

It follows, in particular, that $\|v_t\|_H^2$ is a uniformly continuous bounded function on ρ .

Next, we need a separable subspace of H . By Lemma 2.4 the set v_C is separable in the metric of V . Then v_C is also separable in the metric of H . Its closure in H is a separable Hilbert space, say \hat{H} , and v_C is everywhere dense in \hat{H} in its metric inherited from H .

Now for $t \in \rho$ we define $u_t = v_t$, and if $t \in [0, T] \setminus \rho$, we take any sequence $t_m \rightarrow t$ of elements of ρ such that $v_{t(m)}$ converges weakly in \hat{H} to an element of \hat{H} and call it u_t (we use that $\rho \subset C$ and $v_C \subset \hat{H}$). Observe that, in light of Lemma 2.4, for any $v \in v_C$ and $t \in [0, T]$, we have

$$(u_t, v)_H = (u_0, v)_H + \int_0^t \langle v_s^*, v \rangle ds. \quad (2.6)$$

Since v_C is dense in \hat{H} , this of course implies uniqueness in the definition of u_t for $t \in [0, T] \setminus \rho$ and also implies its weak continuity on $[0, T]$.

Next, take $t \in (0, T]$, $s \in \rho$, $s < t$, sufficiently large k , so that $s \in I_k$ and there are points in I_k which are less than t , call t_k the closest element of I_k to t from the left (it may happen that $t_k = t$) and write that owing to (2.6), for any $v \in v_C$,

$$(u_t - u_{t_k}, v)_H = \int_{t_k}^t \langle v_r^*, v \rangle dr.$$

In light of (1.5)

$$\|u_{t_1}\|_H^2 - \|u_{t_2}\|_H^2 \geq 2\Re(u_{t_1} - u_{t_2}, u_{t_2})_H,$$

implying that (recall that $u_s = v_s$ for $s \in \rho$)

$$\|u_t\|_H^2 - \|u_{t_k}\|_H^2 \geq 2 \int_{t_k}^t \Re \langle v_r^*, v_{\chi_{(-)}(k,r)} \rangle dr.$$

Generally, for two neighboring points $t_1, t_2 \in I_k$ such that $t_1 < t_2$

$$\|u_{t_2}\|_H^2 - \|u_{t_1}\|_H^2 \geq 2 \int_{t_1}^{t_2} \Re \langle v_r^*, v_{\chi_{(-)}(k,r)} \rangle dr,$$

which, similarly to the manipulations after (2.3), leads to

$$\|u_t\|_H^2 - \|u_s\|_H^2 \geq 2 \int_s^t \Re \langle v_r^*, v_{\chi_{(-)}(k,r)} \rangle dr.$$

By what is said before this yields as $k \rightarrow \infty$ that

$$\|u_t\|_H^2 - \|u_s\|_H^2 \geq 2 \int_s^t \Re \langle v_r^*, v_r \rangle dr = \lim_{q \rightarrow t, q \in \rho} \|v_q\|_H^2 - \|v_s\|_H^2.$$

However, since u_t is the weak limit in H of a sequence $v_{t(m)}$ such that $t(m) \in \rho$ and $t(m) \rightarrow t$, we have

$$\|u_t\|_H^2 \leq \varliminf_{m \rightarrow \infty} \|v_{t(m)}\|_H^2.$$

It follows that

$$\|u_t\|_H^2 - \|u_s\|_H^2 = 2 \int_s^t \Re \langle v_r^*, v_r \rangle dr \quad (2.7)$$

as long as $t \in (0, T]$, $s \in \rho$, and $s < t$.

We moved from $t \in (0, T]$ down to $s \in \rho$. Next we move up to s and we only need to do that by taking $t = 0$. So, fix $s \in \rho$, say $s \in I_k$, and denote by s_k the smallest element of I_k . Observe that by (1.5)

$$\|u_0\|_H^2 - \|u_{s_k}\|_H^2 \geq 2\Re(u_0 - u_{s_k}, u_{s_k})_H = -2 \int_0^{s_k} \Re \langle v_r^*, v_{\chi_{(+)}(k,r)} \rangle dr.$$

Generally, for two neighboring points $s_1, s_2 \in I_k$ such that $s_1 < s_2$

$$\|u_{s_1}\|_H^2 - \|u_{s_2}\|_H^2 \geq -2 \int_{s_1}^{s_2} \Re \langle v_r^*, v_{\chi_{(+)}(k,r)} \rangle dr,$$

which again by using telescoping sums leads to

$$\|u_0\|_H^2 - \|u_s\|_H^2 \geq -2 \int_0^s \Re \langle v_r^*, v_{\chi_{(+)}(k,r)} \rangle dr.$$

By letting $k \rightarrow \infty$ we conclude

$$\|u_0\|_H^2 - \|u_s\|_H^2 \geq -2 \int_0^s \Re \langle v_r^*, v_r \rangle dr = \lim_{s \downarrow 0, s \in \rho} \|v_s\|_H^2 - \|u_s\|_H^2.$$

This shows that

$$\|u_0\|_H^2 - \|u_s\|_H^2 = -2 \int_0^s \Re \langle v_r^*, v_r \rangle dr$$

and along with (2.7) yields that, for all $t \in [0, T]$,

$$\|u_t\|_H^2 = \|u_0\|_H^2 + 2 \int_0^t \Re \langle v_r^*, v_r \rangle dr. \quad (2.8)$$

Hence, $\|u_t\|_H^2$ is continuous on $[0, T]$, and, since u_t is weakly continuous, it is strongly continuous on $[0, T]$ as an \hat{H} -valued function and as an H -valued function as well.

Next, we claim that $u_t = v_t$ (a.e.) on $(0, T)$, and, in particular, that one can replace v_r in (2.8) with u_r .

By using Lemma 2.4 and setting $s \downarrow 0$ along ρ in equation (1.2), we get that for all $t \in C$ equation (2.6) holds if we replace u_t in the left-hand side with v_t . Hence,

$$(u_t, v)_H = (v_t, v)_H \quad (2.9)$$

for all $t \in C$ and $v \in v_C$. Furthermore, since $v_C \in \hat{H}$, we have $v_t \in \hat{H}$ if $t \in C$, and $u_t \in \hat{H}$ for all $t \in [0, T]$. In addition v_C is dense in \hat{H} in the metric of \hat{H} . It follows that, for $t \in C$, (2.9) holds for any $v \in \hat{H}$ meaning that $u_t = v_t$ for $t \in C$, that is almost everywhere on $(0, T)$. This proves our claim.

Finally, we deal with (1.3). Since $u_t = v_t$ almost everywhere, for any $v \in V$, we have

$$(u_t, v)_H - (u_s, v)_H = \int_s^t \langle v_r^*, v \rangle dr \quad (2.10)$$

for almost all $s, t \in [0, T]$. Here, for any $v \in V$, the left-hand side is a continuous functions of t, s owing to the strong continuity of u_t in H . It follows that (2.10) holds true for any $s, t \in [0, T]$ and $v \in V$, and this brings the proof of the theorem to an end.

3. A GENERALIZATION

Here we keep Assumption 1.2 and replace Assumptions 1.1 and 1.3 with the following ones. If F, G are Banach spaces, then by $\mathcal{L}(F, G)$ we denote the space of bounded linear operators from F to G .

Assumption 3.1. We have that $\hat{V} := V \cap H$ is dense in H in the metric of H .

We introduce a norm on \hat{V} by $\|\cdot\|_{\hat{V}} = \|\cdot\|_V + \|\cdot\|_H$ which makes $V \cap H$ a Banach space.

Assumption 3.2. There are sequences M_k, M'_k , $k = 1, 2, \dots$, of linear operators $M_k \in \mathcal{L}(V, \hat{V})$ (so to speak, M_k are mollifying operators) and $M'_k \in \mathcal{L}(\hat{V}, V)$ (kind of adjoint to M_k) such that

(i) The norms of operators $M'_k M_k$ as elements of $\mathcal{L}(V, V)$ are bounded by a constant independent of k .

(ii) As $k \rightarrow \infty$, $M_k u \rightarrow u$ in H if $u \in H$, and $M'_k M_k v \rightarrow v$ in V if $v \in V$;

(iii) If $v \in V$, $u \in H$, and $M_k v \rightarrow u$ in H as $k \rightarrow \infty$, then $v = u$.

Assumption 3.3. There is a $w \in H$ such that, for any $k = 1, 2, \dots$ and $v \in V \cap H$, for almost all $t \in [0, T]$, we have

$$(M_k v_t, v)_H = (M_k w, v)_H = \int_s^t \langle v_r^*, M'_k v \rangle dr. \quad (3.1)$$

Theorem 3.4. Under the above assumptions there exists an H -valued H -strongly continuous function u_t , $t \in [0, T]$, such that $u_t = v_t$ for almost all $t \in [0, T]$, $u_0 = w$, and for all $t \in [0, T]$

$$\|u_t\|_H^2 = \|u_0\|_H^2 + 2 \int_0^t \Re \langle v_s^*, u_s \rangle ds. \quad (3.2)$$

Proof. Fix $k = 1, 2, \dots$, and on $v \in \hat{V}$ define $\langle \hat{v}_t^*, v \rangle := \langle v_t^*, M'_k v \rangle$. Observe that $\hat{v}_t^* \in \hat{V}^*$. By applying Corollary 1.5 to \hat{V} , $M_k v_t$, and \hat{v}_t^* in place of V , v_t , and v_t^* , respectively we conclude that there exists an H -valued H -strongly continuous function u_t^k , $t \in [0, T]$, such that $u_t^k = M_k v_t$ for almost all $t \in [0, T]$, $u_0^k = M_k w$, and for all $t \in [0, T]$

$$\|u_t^k\|_H^2 = \|M_k w\|_H^2 + 2 \int_0^t \Re \langle v_s^*, M'_k M_k v_s \rangle ds. \quad (3.3)$$

By applying the same argument to $M_k v_t - M_r v_t$ we see that for all $t \in [0, T]$

$$\|u_t^k - u_t^r\|_H^2 = \|M_k w - M_r w\|_H^2 + 2 \int_0^t \Re \langle v_s^*, M_k' M_k v_s - M_r' M_r v_s \rangle ds. \quad (3.4)$$

It follows that uniformly on $[0, T]$ functions u_t^k converge in H to an H -valued continuous function u_t .

On the set of full measure in $(0, T)$ we have $u_t^k = M_k v_t \rightarrow u_t$ in H . By assumption, $v_t = u_t \in H$ on this set. Passing to the limit in (3.3) on the basis of the dominated convergence theorem presents no difficulty and the theorem is proved.

REFERENCES

- [1] F.E. Browder, *Strongly non-linear parabolic boundary value problems*, American J. of Math, Vol. 86 (1964), No. 2, 339–357.
- [2] N.V. Krylov and B.L. Rozovsky, *Stochastic evolution equations*, “Itogy nauki i tekhniki”, Vol. 14, VINITI, Moscow, 1979, 71-146 in Russian; English translation in J. Soviet Math., Vol. 16 (1981), No. 4, 1233-1277.
- [3] J.-L. Lions, *Espaces intermédiaires entre espaces hilbertiens et applications*, Bull. Math. R.P.R. Bucarest, 2 (1958), 419–432.
- [4] J.-L. Lions, *Équations différentielles opérationnelles dans les espaces de Hilbert*, Equazioni differenziale astratte, Varenna, Italy, 1963, Lectures given at Centro Internazionale Matematico Estivo [held in Varenna (Como), Italy, May 30-June 8 1963], pp. 47- 122, Springer, 2011,
- [5] J.-L. Lions, “Quelques méthodes de résolution des problèmes aux limites non linéaires”, Dunod, Gauthier Villars, 1969.
- [6] E. Pardoux, *Équations aux dérivées partielles stochastiques non linéaires monotones*, Ph. D. Thesis, Université de Paris Sud, Orsay, 1975, http://www.cmi.univ-mrs.fr/~pardoux/Pardoux_these.pdf

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